



TWO APPLICATIONS OF THE BELL POLYNOMIALS

R. Sivaraman*, J. D. Bulnes & J. López-Bonilla*****

* Department of Mathematics, Dwaraka Doss Goverdhan Doss Vaishnav College,
Chennai, Tamil Nadu, India

** Department of Exact Sciences and Technology, Federal University of Amapá, Rod.
Juscelino Kubitschek, Jardim Marco Zero, 68903-419, Macapá, AP, Brazil

*** ESIME-Zacatenco, National Polytechnic Institute, Building 4, 1st Apartment, Col.
Lindavista CP 07738, CDMX, Mexico

Cite This Article: R. Sivaraman, J. D. Bulnes & J. López-Bonilla, "Two Applications of the Bell Polynomials", Indo American Journal of Multidisciplinary Research and Review, Volume 8, Issue 2, July - December, Page Number 120-121, 2024.

Copy Right: © IAJMRR Publication, 2024 (All Rights Reserved). This is an Open Access Article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract:

We show that the Hermite polynomials and the number of self-conjugate permutations of $\{1, 2, \dots, n\}$ can be written in terms of Bell polynomials.

Key Words: Recurrence Relations, Complete Bell Polynomials, Self-Conjugate Permutations, Hermite Polynomials.

1. Introduction:

In [1, 2] we find an expression for the complete Bell polynomials:

$$B_m(x_1, x_2, \dots, x_m) = \begin{vmatrix} \binom{m-1}{0} x_1 & \binom{m-1}{1} x_2 & \dots & \binom{m-1}{m-2} x_{m-1} & \binom{m-1}{m-1} x_m \\ -1 & \binom{m-2}{0} x_1 & \dots & \binom{m-2}{m-3} x_{m-2} & \binom{m-2}{m-2} x_{m-1} \\ 0 & -1 & \dots & \binom{m-3}{m-4} x_{m-3} & \binom{m-3}{m-3} x_{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \binom{1}{0} x_1 & \binom{1}{1} x_2 \\ 0 & 0 & \dots & -1 & \binom{0}{0} x_1 \end{vmatrix}, \quad (1)$$

Therefore:

$$B_0 = 1, B_1 = x_1, B_2 = x_1^2 + x_2, B_3 = x_1^3 + 3x_1 x_2 + x_3, B_4 = x_1^4 + 6x_1^2 x_2 + 4x_1 x_3 + 3x_2^2 + x_4, \quad (2)$$

$$B_5 = x_1^5 + 10x_1^3 x_2 + 10x_1^2 x_3 + 15x_1 x_2^2 + 5x_1 x_4 + 10x_2 x_3 + x_5, \dots$$

On the other hand, from [3] it is clear that the recurrence relation:

$$y_n = \sum_{j=1}^n \binom{n-1}{j-1} x_j y_{n-j}, \quad n \geq 1, \quad (3)$$

Implies the following property in terms of Bell polynomials:

$$y_n = B_n(x_1, x_2, \dots, x_n), \quad (4)$$

That is, (4) is the solution of (3).

The Hermite polynomials $H_n(x)$ [4-11] and U_n , the number of self-conjugate permutations of $\{1, 2, \dots, n\}$ [12], verify recurrence relations with the structure (3), hence they can be written in the form (4), which is showed in Sec. 2.

2. Two Applications of (3) and (4):

The Hermite polynomials satisfy the recurrence relation [4-11]:

$$H_n(x) = 2x H_{n-1}(x) - 2(n-1) H_{n-2}(x), H_1(x) = 1, H_2(x) = 2x, \quad n \geq 2, \quad (5)$$

In harmony with (3) for $x_1 = 2x, x_2 = -2, x_k = 0, k \geq 3$, then from (4):

$$H_n(x) = B_n(2x, -2, 0, \dots, 0), \quad n \geq 0, \quad (6)$$

Thus:

$$H_0 = B_0 = 1, H_1 = B_1(2x) = 2x, H_2 = B_2(2x, -2) = 4x^2 - 2, H_3 = B_3(2x, -2, 0) = 8x^3 - 12x, \quad (7)$$

$$H_4 = B_4(2x, -2, 0, 0) = 16x^4 - 48x^2 + 12, \dots$$

Verifying the differential equation:

$$H_n'' - 2x H_n' + 2n H_n = 0. \quad (8)$$

In [12] we find the following explicit formula for the number of self-conjugate permutations of $\{1, \dots, n\}$:

$$U_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n!}{2^k k! (n-2k)!}, \quad n \geq 0, \quad (9)$$

Such that:

$$U_n = U_{n-1} + (n-1) U_{n-2}, \quad U_0 = 1, U_1 = 1, \quad n \geq 2, \quad (10)$$

Which it is compatible with (3) for $x_1 = 1, x_2 = 1, x_k = 0, k \geq 3$, then (4) gives the property:

$$U_n = B_n(1, 1, 0, \dots, 0), \quad n \geq 0, \quad (11)$$

Thus $U_0 = U_1 = 1, U_2 = 2, U_3 = 4, U_4 = 10, U_5 = 26, \dots$

In [3, 13-15] there are more applications of the complete Bell polynomials.

References:

1. W. P. Johnson, The curious history of Faà di Bruno's formula, The Math. Assoc. of America 109 (2002), 217-234.
2. J. López-Bonilla, R. López-Vázquez, S. Vidal-Beltrán, Bell polynomials, Prespacetime9, No. 5 (2018), 451-453.
3. R. Sivaraman, J. D. Bulnes, J. López-Bonilla, Complete Bell polynomials and recurrence relations for arithmetic functions, European J. of Theor. & Appl. Sci. 1, No. 3 (2023) 167-170
4. Ch. Hermite, Sur un nouveau développement en série de fonctions, Compt. Rend. Acad. Sci. Paris 58, (1864) 93-100 & 266-273.
5. M. Abramowitz, I. A. Stegun, Handbook of mathematical functions, Wiley & Sons, New York (1972).
6. C. Lanczos, Applied analysis, Dover, New York (1988).
7. J. López-Bonilla, A. Lucas-Bravo, S. Vidal-Beltrán, Integral relationship between Hermite and Laguerre polynomials: Its application in quantum mechanics, Proc. Pakistan Acad. Sci. 42, No. 1 (2005) 63-65.
8. C. T. Aravnis, Hermite polynomials and quantum harmonic oscillator, B. S. Math. Exchange 7 (2010), 27-30.
9. J. López-Bonilla, G. Posadas-Durán, On the Saha's generating function for the Hermite polynomials, Prespacetime Journal 7, No. 13 (2016) 1805-1806.
10. S. Álvarez-Ballesteros, J. López-Bonilla, H. Torres-Silva, Cohen's formula applied to Laplace-Chebyshev-Hermite polynomials, Prespacetime J. 8, No. 1 (2017) 93-95.
11. J. López-Bonilla, S. Vidal-Beltrán, On the Hermite polynomials, Comput. And Appl. Math. Sci. 6, No. 2 (2021) 33-34.
12. W. P. Johnson, An introduction to q-analysis, Am. Math. Soc., Providence, Rhode Island, USA (2020).
13. J. López-Bonilla, S. Vidal-Beltrán, A. Zúñiga-Segundo, Some applications of complete Bell polynomials, World Eng. & Appl. Sci. J. 9, No. 3 (2018) 89-92.
14. J. López-Bonilla, J. YaljáMontiel-Pérez, S. Vidal-Beltrán, Gamma function, Bernoulli numbers and Bell polynomials, African J. of Basic & Appl. Sci. 11, No. 2 (2019) 25-26.
15. J. López-Bonilla, S. Vidal-Beltrán, Bell polynomials and derivatives of gamma function, Studies in Nonlinear Sci. 4, No. 1 (2019) 3-4.